

If f is continuous and $\int_3^8 f(x) dx = -14$, find $\int_2^3 x f(12 - x^2) dx$.

SCORE: / 12 PTS

④ $u = 12 - x^2 \begin{cases} x=3 \rightarrow u=3 \\ x=2 \rightarrow u=8 \end{cases}$

$$\frac{du}{dx} = -2x$$

$$dx = -\frac{1}{2x} du$$

$$x f(12 - x^2) dx = x f(u) \cdot -\frac{1}{2x} du = -\frac{1}{2} f(u) du$$

① $\int_8^3 -\frac{1}{2} f(u) du = \frac{1}{2} \int_3^8 f(u) du$
 ④ $= \frac{1}{2} (-14)$
 $= -7$
 ②

Find $\lim_{n \rightarrow \infty} \frac{21}{n} \sum_{i=1}^n \frac{1}{\sqrt[3]{1 + \frac{7i}{n}}}$ by finding the corresponding definite integral, and evaluating that integral.

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LET $1 + \frac{7i}{n} = a + i\Delta x$

$a = 1, \Delta x = \frac{7}{n} = \frac{b-a}{n} \rightarrow b = 8$
 ③ $\frac{21}{n} = 3\Delta x$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{\sqrt[3]{1 + \frac{7i}{n}}} \Delta x = \int_1^8 \frac{3}{\sqrt[3]{x}} dx = \int_1^8 3x^{-\frac{1}{3}} dx = \frac{9}{2} x^{\frac{2}{3}} \Big|_1^8$$

LET $f(x) = \frac{3}{\sqrt[3]{x}}$

$= \frac{9}{2} (4 - 1) = \frac{27}{2}$ ②

The table below gives the rate $p(t)$ at which the pressure of a tank of gas increased as the temperature increased (measured in kPa per degree Celsius). When the temperature was 22° Celsius, the pressure was 180 kPa.

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t	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
$p(t)$	1.0	1.2	1.4	1.7	2.0	2.3	2.7	3.1	3.5	3.9	4.4	4.9	5.5	6.2	7.0	7.9	9.0

[a] Write an expression (involving an integral) for the pressure when the temperature was 46° Celsius.

⑤ LET $P(t)$ = PRESSURE WHEN TEMPERATURE IS $t^\circ\text{C}$, so $P'(t) = p(t)$
 $\int_{22}^{46} P'(t) dt = P(46) - P(22) \rightarrow P(46) = 180 + \int_{22}^{46} p(t) dt$

[b] Estimate the pressure when the temperature was 46° Celsius, using the answer to part [a], 3 subintervals and the Midpoint Rule.

⑤ $\Delta t = \frac{46-22}{3} = 8$ SUBINTERVALS = $[22, 30], [30, 38], [38, 46]$
 MIDPOINTS = 26, 34, 42

$180 + (p(26) + p(34) + p(42)) \Delta t$ ③
 $= 180 + (2.0 + 3.5 + 5.5) 8$ ③
 $= 180 + 11 * 8 = 180 + 88 = 268 \text{ kPa}$ ①

③

Answer the following questions about the definition of the definite integral as presented in lecture.

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(Your answers may refer to the fact that the definite integral equals the area under a curve which is above the x -axis.)

[a] What does the formula inside the summation ($f(x_i^*) \Delta x$) represent?

AREA OF EACH RECTANGLE USED TO APPROXIMATE AREA

[b] What is the difference between using $f(x_i^*)$ and $f(a + i\Delta x)$ in the definition?

x_i^* MEANS ANY POINT IN EACH SUBINTERVAL

$a + i\Delta x$ MEANS ONLY ENDPOINTS

Evaluate the following integrals.

SCORE: ____ / 50 PTS

[a] $\int_{-3}^3 (8\sqrt{9-t^2} - \frac{2t^3}{4+3t^2}) dt$

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$= 8 \int_{-3}^3 \sqrt{9-t^2} dt - \int_{-3}^3 \frac{2t^3}{4+3t^2} dt$

$= 8 * \text{AREA OF SEMI-CIRCLE OF RADIUS 3}$

$= 8 * \frac{1}{2} (9\pi)$

$= 36\pi$

$\frac{2(-t)^3}{4+3(-t)^2} = -\frac{2t^3}{4+3t^2}$

2ND INTEGRAND IS ODD, CONTINUOUS

INTEGRAL = 0

[c] $\int_0^{\frac{\pi}{3}} \frac{\sin 2x}{1+\cos^2 x} dx$

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$u = 1 + \cos^2 x$ $\begin{cases} x = \frac{\pi}{3} \rightarrow u = \frac{5}{4} \\ x = 0 \rightarrow u = 2 \end{cases}$

$\frac{du}{dx} = 2\cos x (-\sin x)$

$\frac{du}{dx} = -2\sin 2x$

$dx = \frac{du}{-2\sin 2x}$

$\frac{\sin 2x}{1+\cos^2 x} dx = \frac{\sin 2x}{1+\cos^2 x} \frac{du}{-2\sin 2x} = -\frac{1}{2} \frac{1}{u} du$

$\int_2^{\frac{5}{4}} -\frac{1}{2} \frac{1}{u} du = -\frac{1}{2} \ln|u| \Big|_2^{\frac{5}{4}} = -\frac{1}{2} (\ln \frac{5}{4} - \ln 2) = -\frac{1}{2} \ln \frac{5}{8} = \frac{1}{2} \ln \frac{8}{5}$

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$\int \frac{1-x}{\sqrt{1-2x}} dx$

SEE ALSO ALTERNATE SOLUTION

③ $u = 1-2x \rightarrow x = \frac{1-u}{2}$

$\frac{du}{dx} = -2 \rightarrow dx = -\frac{1}{2} du$

$\frac{1-x}{\sqrt{1-2x}} dx = \frac{1-x}{\sqrt{1-2x}} \cdot -\frac{1}{2} du$

$= \frac{1-\frac{1-u}{2}}{\sqrt{u}} \cdot -\frac{1}{2} du$

$= -\frac{1+u}{4\sqrt{u}} du$

$\int (-\frac{1}{4} u^{-\frac{1}{2}} - \frac{1}{4} u^{\frac{1}{2}}) du$

$= -\frac{1}{2} u^{\frac{1}{2}} - \frac{1}{6} u^{\frac{3}{2}} + C$

$= -\frac{1}{2} (1-2x)^{\frac{1}{2}} - \frac{1}{6} (1-2x)^{\frac{3}{2}} + C$

ALTERNATE SOLUTION

$$[b] \int \frac{1-x}{\sqrt{1-2x}} dx$$

$$\textcircled{4} u = \sqrt{1-2x} \rightarrow x = \frac{1-u^2}{2}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{1-2x}} \cdot (-2)$$

$$dx = -\sqrt{1-2x} du$$

$$\frac{1-x}{\sqrt{1-2x}} dx = \frac{1-x}{\sqrt{1-2x}} \cdot -\sqrt{1-2x} du$$

$$= (x-1) du$$

$$= \left(\frac{1-u^2}{2} - 1 \right) du$$

$$= \left[-\frac{1+u^2}{2} du \right] \textcircled{5}$$

$$\int -\frac{1+u^2}{2} du = \left[-\frac{1}{2}u - \frac{1}{6}u^3 + C \right] \textcircled{3}$$

$$= \left[-\frac{1}{2}\sqrt{1-2x} - \frac{1}{6}(\sqrt{1-2x})^3 + C \right]$$

$\textcircled{3}$

$\textcircled{1}$

Find $\frac{d}{dx}(\cosh^{-1} 2x + \operatorname{sech} x^2)$.

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You may use any hyperbolic identities or the derivatives of any hyperbolic functions without proving them.

$$\begin{aligned} & \textcircled{2\frac{1}{2}} \left[\frac{1}{\sqrt{(2x^2-1)}} \right] \cdot \underbrace{2}_{\textcircled{2}} - \underbrace{\text{sech } x^2 \tanh x^2}_{\textcircled{2\frac{1}{2}}} \cdot \underbrace{2x}_{\textcircled{2}} \\ &= \frac{2}{\sqrt{4x^2-1}} - 2x \text{sech } x^2 \tanh x^2 \quad \textcircled{+1} \text{ SIMPLIFIED} \end{aligned}$$

The graph of $f(t)$ is shown on the right. NOTE: The graph consists of 7 straight line segments.

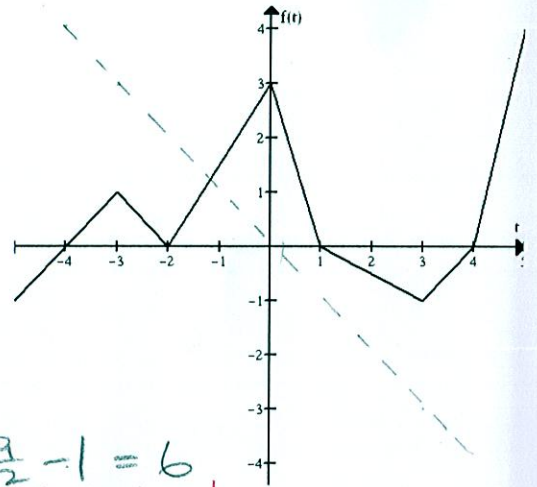
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Let $g(x) = \int_{-2}^x (t + f(t)) dt$.

[a] Find $g(3)$.

(15) $\int_{-2}^3 (t + f(t)) dt$
 $= \int_{-2}^3 t dt + \int_{-2}^3 f(t) dt$ (3)
 $= \frac{1}{2} t^2 \Big|_{-2}^3 + \int_{-2}^1 f(t) dt + \int_1^3 f(t) dt$
 $= \frac{1}{2}(9-4) + \frac{1}{2}(3)(3) - \frac{1}{2}(2)(1) = \frac{5}{2} + \frac{9}{2} - 1 = 6$
 (3) (3) (3) (1)

(b) Find $\sigma'(3)$



[b] Find $g'(3)$.

8. $g'(x) = x + f(x)$ ③
 $g'(3) = 3 + f(3) = 3 + -1 = 2$
② ② ①

[c] Find $g''(-1)$.

⑧ $g'(x) = 1 + f'(x)$ ③
 $g'(-1) = 1 + f'(-1) = 1 + \frac{3}{2} = \frac{5}{2}$
 ② ② ①

[d] OPTIONAL BONUS QUESTIONS:

BONUS SCORE: _____ / 15 PTS

[i] Estimate all critical number(s) of g .

$$g'(x) = 0 \rightarrow \underbrace{x + f(x) = 0}_{(1)} \rightarrow \underbrace{f(x) = -x}_{x \approx -1\frac{1}{3}} \quad (2)$$

[ii] Find all inflection points of g .

② $g''(x)$ CHANGES SIGNS @ $x = 0, 1$
 $1 + f'(x) > 0 \rightarrow f'(x) > -1$ ON $[-5, -3] [-2, 0] [1, 5]$
 $1 + f'(x) < 0 \rightarrow f'(x) < -1$ ON $[0, 1]$